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The adapted solution and comparison theorem for backward stochastic differential equations with Poisson jumps and applications

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ABSTRACT

This paper deals with a class of backward stochastic differential equations with Poisson jumps and with random terminal times. We prove the existence and uniqueness result of adapted solution for such a BSDE under the assumption of non-Lipschitzian coefficient. We also derive two comparison theorems by applying a general Girsanov theorem and the linearized technique on the coefficient. By these we first show the existence and uniqueness of minimal solution for one-dimensional BSDE with jumps when its coefficient is continuous and has a linear growth. Then we give a general Feynman–Kac formula for a class of parabolic types of second-order partial differential and integral equations (PDIEs) by using the solution of corresponding BSDE with jumps. Finally, we exploit above Feynman–Kac formula and related comparison theorem to provide a probabilistic formula for the viscosity solution of a quasi-linear PDIE of parabolic type.

1. Introduction

Suppose that $W_t^T = (W_t^1, \dots, W_t^d)$, $t \geqslant 0$, is a d-dimensional Brownian motion; $K^T(t) = (K_1(t), \dots, K_r(t))$, $t \geqslant 0$, is an r-dimensional stationary Poisson point process taking values in a measurable space $(E, \mathcal{B}(E))$. We denote by $N_{K_i}(ds, dx)$, $i = 1, \dots, r$, the Poisson counting measure induced by $K_i(\cdot)$ with compensator $\lambda(dx)\,ds$, and by $\widetilde{N}_{K_i}(ds, dx)$ the martingale measure such that $\widetilde{N}_{K_i}(ds, dx) = N_{K_i}(ds, dx) - \lambda(dx)\,ds$, where $\lambda(\cdot)$ is a σ-finite measure on $\mathcal{B}(E)$.

Let $(\Omega, \mathcal{F}, P)$ be a complete and standard measurable probability space (cf. N. Ikeda and S. Watanabe [7]) equipped with a filtration denoted by $\mathcal{F}_t = \sigma[W_s;\ s \leqslant t] \vee \sigma[N_K((0,s], A);\ s \leqslant t,\ A \in \mathcal{B}(E)] \vee \mathcal{N}$, $0 \leqslant t \leqslant \tau$, where $\mathcal{N}$ is the all P-null sets; τ is a stopping time. The following notation will be used in this paper.

- $S^2_{\mathcal{F}}(0, \tau; R^n)$ denotes the set of all $\mathcal{F}_t$-adapted RCLL processes $X(\cdot)$ valued in R^n such that $E[\sup_{0 \leqslant t \leqslant \tau} |X(t)|^2] < \infty$;
- $L^2_{\mathcal{F}}(0, \tau; R^n(R^{n\otimes d}))$ denotes the set of all $\mathcal{F}_t$-adapted processes $Y(\cdot)$ valued in $R^n(R^{n\otimes d})$ such that $E\int_0^\tau |Y(t)|^2\,dt < \infty$;
- $H^2_{\mathcal{F}}(0, \tau; R^{n\otimes r})$ denotes the set of all $\mathcal{P} \otimes \mathcal{B}(E)$ measurable processes $\Psi_t(x)$ valued in $R^{n\otimes r}$ such that

$$E\left[\int_0^\tau \int_E \left|\Psi_t(x)\right|^2 \lambda(dx)\,dt\right] := E\int_0^\tau \left\|\Psi_t(\cdot)\right\|^2 dt < \infty,$$

where $\mathcal{P}$ is the σ-algebra generated by all predictable subsets of $\Omega \times [0, \tau]$;

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- $L^2_\lambda(0,\tau;R^{n\otimes r})$ denotes the set of all $\mathcal{B}(E)$-measurable functions $\Phi(\cdot)$ valued in $R^{n\otimes r}$ such that

$$\|\Phi\|^2=\int_E \left|\Phi(x)\right|^2\lambda(dx)<\infty.$$

We shall consider the following backward stochastic differential equation with Poisson jumps and with a random terminal time (BSDE for short):

$$Y_t=\xi+\int_{t\wedge\tau}^{\tau} f(s,Y_s,Z_s,U_s,\omega)\,ds-\int_{t\wedge\tau}^{\tau} Z_s\,dW_s-\int_{t\wedge\tau}^{\tau}\int_E U_s(x)\widetilde{N}_K(ds,dx), \tag{1.1}$$

where the random function $f:[0,\infty)\times R^n\times R^{n\otimes d}\times L^2_\lambda(R^{n\otimes r})\times\Omega\to R^n$ is jointly measurable and $\mathcal{F}_t$-adapted; ξ is $\mathcal{F}_\tau$-measurable and square integrable. If there exists a triplet $(Y,Z,U)\in S^2_{\mathcal{F}}(0,\tau;R^n)\times L^2_{\mathcal{F}}(0,\tau;R^{n\otimes d})\times H^2_{\mathcal{F}}(0,\tau;R^{n\otimes r})$ satisfying (1.1), we call it an adapted solution to such a BSDE.

In order to guarantee existence and uniqueness of the adapted solution to above BSDE, we have to impose some assumptions on the coefficient f. The reader is referred to El Karoui and Mazliak [5] for the general theory of continuous BSDEs, which are only driven by a multidimensional Brownian motion, to Tang and Li [20], Situ [19], Barles, Buckdahn and Pardoux [1], Yin and Situ [21], Yin and Wang [22] for the general theory and some applications of BSDEs with Poisson jumps.

The comprehensive applications of BSDEs have motivated many efforts to establish the existence and uniqueness of adapted solution under general hypotheses on the coefficient. For instance, Peng [14] first introduced monotonic coefficient and Mao [10] discussed non-Lipschitz coefficient for multidimensional continuous BSDEs. For the one-dimensional case, Lepeltier and San Martin [8] proved the existence of a solution to continuous BSDEs with the help of comparison theorem under the assumption of continuous coefficient. Besides, Pardoux [12] considered multidimensional BSDEs with jumps and showed an existence theorem of a solution under the monotonicity condition w.r.t. Y and the Lipschitz condition w.r.t. (Z,U). This result is also used by Royer [17] for improving a comparison theorem derived by Barles et al. [1].

The comparison theorem is also an important property of BSDEs. We refer to El Karoui, Peng, Quenez [6] for their applications to finance, to Peng [14,15] for the applications to stochastic control and to Peng [16], Coquet et al. [4] for the applications to nonlinear expectations. There are much works involving the comparison theorem of continuous BSDEs, see for instance, Darling and Pardoux [3], Liu and Ren [9] and the above references. However, there are only a few important results concerned with comparison theorem for BSDEs with jumps. We shall mention here Barles et al. [1] and Pardoux [12], who use the comparison theorem of BSDEs with jumps to solve integral–partial differential equations and to study nonlinear expectations and nonlinear martingales with jumps.

The aim of this paper is to study BSDEs with Poisson jumps and random terminal times and to consider their applications. The first result of this paper is to establish the existence and uniqueness theorem of adapted solution to Eq. (1.1). The proof is based on the smoothing technique, which makes our conditions on the coefficient f to be weaker than those of Pardoux [12] and generalizes his result. We then prove two comparison theorems of BSDEs with jumps. It should be noted that Corollary 3.1 of this paper is similar to Theorem 2.6 in Royer [17]. However, compared with her result, our conditions on Y and Z are weaker, and indeed ours improves the existing result due to random terminal time, r-dimensional Poisson point process and the important fact that this paper allows $c\geqslant -1$ but not $c>-1$ as C_1 in Theorem 2.6 of [17].

This paper is organized as follows: in Section 2, we give an existence and uniqueness result of adapted solution to Eq. (1.1), which also generalizes the corresponding result in Situ and the first author [21]. In Section 3, we prove two comparison theorems of BSDEs by means of a general Girsanov theorem and a linearized treatment on the coefficients. In Section 4, we focus on some applications of the comparison theorem of BSDEs. Firstly, we prove the existence and uniqueness of minimal solution for one-dimensional BSDE with Poisson jumps under the assumptions that the coefficient is continuous and has a linear growth. Then we give a probabilistic formula for a class of quasi-linear second-order partial differential and integral equations (PDIEs) by using the solutions of BSDEs with Poisson jumps, which is called a general Feynman–Kac formula. Finally, we exploit above Feynman–Kac formula and related comparison theorem to provide a probabilistic formula for the viscosity solution of a quasi-linear PDIE of parabolic type.

2. Existence and uniqueness of adapted solutions

This section will present two existence and uniqueness results of adapted solutions to BSDEs with Poisson jumps and with random terminal times. The first one, that is Lemma 2.1, due to Situ and the first author [21], is needed in our subsequent discussion.

Lemma 2.1. *Suppose that the following conditions hold*:

(i) $E(\int_0^\tau |f(t,0,0,0,\omega)|\,dt)^2<\infty$, $\xi\in\mathcal{F}_\tau$, $E|\xi|^2<\infty$;
(ii) *For any* $Q_i=(Y_i,Z_i,U_i)\in R^n\times R^{n\otimes d}\times L^2_\lambda(R^{n\otimes r})$, $i=1,2$, $t\geqslant 0$,

$$\left|f(t,Q_1,\omega)-f(t,Q_2,\omega)\right|\leqslant u_1(t)|Y_1-Y_2|+u_2(t)\left[|Z_1-Z_2|+\|U_1-U_2\|\right],$$

where $u_1(t)$ and $u_2(t)$ are nonnegative, deterministic functions and satisfy

$$\int_0^\infty u_1(t)\,dt + \int_0^\infty u_2(t)^2\,dt < \infty. \tag{2.1}$$

Then (1.1) *has a unique solution.*

The next theorem generalizes the result of Lemma 2.1 to the case where f is continuous but not Lipschitz continuous. We impose some assumptions as follows:

(H1) $f(t,Y,Z,U,\omega) = f_1(t,Y,Z,U,\omega) + f_2(t,Y,Z,U,\omega)$;
(H2) $f_1(t,Y,Z,U,\omega)$ is continuous in (Y,Z,U). Moreover,

$$\left|f_1(t,Y,Z,U,\omega)\right| \leqslant u_1(t)\big(1+|Y|\big), \tag{2.2}$$

$$\begin{aligned} &(Y_1-Y_2)\cdot\big(f_1(t,Y_1,Z_1,U_1,\omega) - f_1(t,Y_2,Z_2,U_2,\omega)\big)\\ &\quad\leqslant u_1(t)\rho\big(|Y_1-Y_2|^2\big) + u_2(t)|Y_1-Y_2|\big(|Z_1-Z_2| + \|U_1-U_2\|\big), \end{aligned} \tag{2.3}$$

$$\left|f_1(t,Y,Z,U_1,\omega) - f_1(t,Y,Z,U_2,\omega)\right| \leqslant u_2(t)\|U_1-U_2\|; \tag{2.4}$$

(H3) $|f_2(t,0,0,0,\omega)| \leqslant u_1(t)$, and for $Q_i = (Y_i,Z_i,U_i)$, $i=1,2$,

$$\left|f_2(t,Q_1,\omega) - f_2(t,Q_2,\omega)\right| \leqslant u_1(t)|Y_1-Y_2| + u_2(t)\big(|Z_1-Z_2| + \|U_1-U_2\|\big);$$

(H4) $\xi \in \mathcal{F}_\tau$, $E|\xi|^2 < \infty$;

where $u_1(t)$ and $u_2(t)$ satisfy (2.1); $\rho(u)$ is a nondecreasing, continuous and concave function from R_+ to R_+ such that $\rho(0)=0$, $\rho(u)\geqslant 0$, for $u \geqslant 0$ and $\int_{0+}\frac{du}{\rho(u)} = \infty$.

Theorem 2.1. *Under assumptions of* (H1)–(H4), (1.1) *has a unique solution.*

The proof of Theorem 2.1 will be given in an appropriate position. As a preparation, we first derive a priori estimate.

Lemma 2.2. *Let* (2.2) *of* (H2) *and* (H3) *be satisfied. Assume that there exists a constant $k_0 \geqslant 0$ such that $|\xi| \leqslant k_0$ a.s. If (Y,Z,U) is a solution of* (1.1)*, then for each $t\in[0,\tau]$,*

$$|Y_t| \leqslant N_0, \quad P\text{-a.s.},$$

where $N_0 \geqslant 0$ is a constant depending on $\int_0^\infty (u_1(t)+u_2(t)^2)\,dt$ and k_0 only.

Proof. By Itô's formula, we can obtain that $(Q_s := (Y_s,Z_s,U_s))$

$$|Y_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{\tau} |Z_s|^2\,ds + \int_{t\wedge\tau}^{\tau}\int_E \left|U_s(x)\right|^2 N_K(ds,dx) = |\xi|^2 + \int_{t\wedge\tau}^{\tau} 2Y_s\cdot f(s,Q_s,\omega)\,ds - \int_{t\wedge\tau}^{\tau} dM_s,$$

where

$$M_t = 2\int_0^t Y_s\cdot Z_s\,dW_s + 2\int_0^t\int_E Y_s\cdot U_s(x)\widetilde{N}_K(ds,dx)$$

is a uniformly integrable martingale from the Burkhoder–Davis–Gundy inequality. By this, (H3) and (2.2), together with elementary algebraic inequality, one obtains that

$$\begin{aligned} |Y_{t\wedge\tau}|^2 &\leqslant E^{\mathcal{F}_{t\wedge\tau}}\left[|Y_{t\wedge\tau}|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau}|Z_s|^2\,ds + \frac{1}{2}\int_{t\wedge\tau}^{\tau}\|U_s\|^2\,ds\right]\\ &\leqslant E^{\mathcal{F}_{t\wedge\tau}}\left[|\xi|^2 + 2\int_0^\infty u_1(s)\,ds + \int_{t\wedge\tau}^{\tau}\big(6u_1(s)+4u_2(s)^2\big)|Y_s|^2\,ds\right]\\ &\leqslant k_0^2 + 2\int_0^\infty u_1(s)\,ds + \int_t^\infty \big(6u_1(s)+4u_2(s)^2\big)E^{\mathcal{F}_{t\wedge\tau}}|Y_{s\wedge\tau}|^2\,ds, \end{aligned}$$

where $E^{\mathcal{F}_t}[\xi] = E[\xi|\mathcal{F}_t]$. By virtue of Gronwall's lemma, we can easily get

$$|Y_{t\wedge\tau}|^2 \leqslant \left(k_0^2 + 2\int_0^\infty u_1(s)\,ds\right)\exp\left(\int_0^\infty \left(6u_1(s) + 4u_2(s)^2\right)ds\right) := N_0. \qquad \square$$

Proof of Theorem 2.1. *Uniqueness*: Let (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) be two solutions of (1.1). Then applying Itô's formula to $|Y_t^1 - Y_t^2|^2$, we have

$$\begin{aligned} A_t &:= E\left|Y_{t\wedge\tau}^1 - Y_{t\wedge\tau}^2\right|^2 + \frac{1}{2}E\int_{t\wedge\tau}^{\tau}\left|Z_s^1 - Z_s^2\right|^2 ds + \frac{1}{2}E\int_{t\wedge\tau}^{\tau}\left\|U_s^1 - U_s^2\right\|^2 ds \\ &\leqslant E\int_{t\wedge\tau}^{\tau}\left[\left(2u_1(s) + 16u_2(s)^2\right)\rho_1\left(\left|Y_s^1 - Y_s^2\right|^2\right)\right]ds \\ &= E\int_t^\infty\left[\left(2u_1(s) + 16u_2(s)^2\right)\rho_1\left(\left|Y_{s\wedge\tau}^1 - Y_{s\wedge\tau}^2\right|^2\right)\right]ds \\ &\leqslant \int_t^\infty\left[\left(2u_1(s) + 16u_2(s)^2\right)\rho_1(A_s)\right]ds, \end{aligned}$$

where $\rho_1(u) = \rho(u) + u$ has the same property as $\rho(u)$. Therefore we have $A_t = 0$, $t \in [0, \tau]$, by the Bahari inequality (see Mao [11, pp. 45–46] or Situ [18]), and the uniqueness follows.

Existence: The proof will be divided into three steps.

Step 1. Assume that (H1)–(H4) hold except that (2.2) is replaced with the following

$$\left|f_1(t, Y, Z, U, \omega)\right| \leqslant u_1(t). \tag{2.5}$$

Let

$$f_1^n(t, Y, Z, U) = \int_{R^n\times R^{n\otimes d}} f_1\left(t, Y - n^{-1}\overline{Y}, Z - n^{-1}\overline{Z}, U\right)J(\overline{Y}, \overline{Z})\,d\overline{Y}\,d\overline{Z},$$

where $J(Y, Z) = J_1(Y)J_2(Z)$, and $J_1(Y)$ is defined, for all $Y \in R^n$, as

$$J_1(Y) = \begin{cases} C_1\exp(-(1 - |Y|^2)^{-1}) & \text{for } |Y| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

such that the constant C_1 satisfies $\int_{R^n} J_1(Y)\,dY = 1$. $J_2(Z)$ is similarly defined for any $Z \in R^{n\otimes d}$, which can be viewed as an $(n\cdot d)$-dimensional vector. It is easy to check that $f^n(t, Y, Z, U) = f_1^n(t, Y, Z, U) + f_2(t, Y, Z, U)$ satisfies (i) and (ii) of Lemma 2.1 but with different $u_1(t)$ and $u_2(t)$, which depend on n and satisfy (2.1). Hence by Lemma 2.1 the BSDE

$$Y_t^n = \xi + \int_{t\wedge\tau}^{\tau} f^n\left(s, Y_s^n, Z_s^n, U_s^n\right)ds - \int_{t\wedge\tau}^{\tau} Z_s^n\,dW_s - \int_{t\wedge\tau}^{\tau}\int_E U_s^n(x)\widetilde{N}_K(ds, dx) \tag{2.6}$$

has a unique solution (Y^n, Z^n, U^n). We first note, from (2.5) and (H3), that

$$\left|f^n(t, Y, Z, U)\right| \leqslant 2u_1(t) + u_1(t)|Y| + u_2(t)\left[|Z| + \|U\|\right].$$

Therefore, in analogy with Lemma 1 in Situ [19], we can easily deduce that

$$\sup_n E\left(\sup_{0\leqslant t\leqslant\tau}\left|Y_t^n\right|^2 + \int_0^\tau\left|Z_t^n\right|^2 dt + \int_0^\tau\left\|U_t^n\right\|^2 dt\right) \leqslant k_1 < \infty, \tag{2.7}$$

where $k_1 \geqslant 0$ is a constant only depending on $\int_0^\infty (u_1(t) + u_2(t)^2)\,dt$ and $E|\xi|^2$.

Applying Itô's formula to $|Y_t^m - Y_t^n|^2$ yields that

$$E\left(\left|Y_{t\wedge\tau}^n - Y_{t\wedge\tau}^m\right|^2 + \int_{t\wedge\tau}^{\tau} \left|Z_s^n - Z_s^m\right|^2 ds + \int_{t\wedge\tau}^{\tau} \left\|U_s^n - U_s^m\right\|^2 ds\right)$$
$$= 2E\int_{t\wedge\tau}^{\tau} \left(Y_s^n - Y_s^m\right)\cdot\left(f^n\left(s, Y_s^n, Z_s^n, U_s^n\right) - f^m\left(s, Y_s^m, Z_s^m, U_s^m\right)\right) ds$$
$$\leqslant 2E\int_{t\wedge\tau}^{\tau}\int_{R^n\times R^{n\otimes d}} \left[u_1(s)\rho\left(\left|Y_s^n - Y_s^m - \left(n^{-1} - m^{-1}\right)\overline{Y}\right|^2\right) + u_2(s)\left|Y_s^n - Y_s^m - \left(n^{-1} - m^{-1}\right)\overline{Y}\right|\right.$$
$$\left.\times\left(\left|Z_s^n - Z_s^m - \left(n^{-1} - m^{-1}\right)\overline{Z}\right| + \left\|U_s^n - U_s^m\right\|\right) + 2u_1(s)\left|n^{-1} - m^{-1}\right|\left|\overline{Y}\right|\right] J(\overline{Y}, \overline{Z})\, d\overline{Y}\, d\overline{Z}\, ds$$
$$+ E\int_{t\wedge\tau}^{\tau} \left[2u_1(s)\left|Y_s^n - Y_s^m\right|^2 + 2u_2(s)\left|Y_s^n - Y_s^n\right|\left(\left|Z_s^n - Z_s^m\right| + \left\|U_s^n - U_s^m\right\|\right)\right] ds$$
$$\leqslant k_2 E\int_{t\wedge\tau}^{\tau}\int_{R^n} \left(u_1(s) + u_2(s)^2\right)\rho_1\left(\left|Y_s^n - Y_s^m - \left(n^{-1} - m^{-1}\right)\overline{Y}\right|\right) J_1(\overline{Y})\, d\overline{Y}\, ds$$
$$+ k_2 E\int_{t\wedge\tau}^{\tau} \left(u_1(s) + u_2(s)^2\right)\left|Y_s^n - Y_s^m\right|^2 ds + k_2\left(n^{-1} + m^{-1}\right) + \frac{1}{2}E\int_{t\wedge\tau}^{\tau} \left(\left|Z_s^n - Z_s^m\right|^2 + \left\|U_s^n - U_s^m\right\|^2\right) ds,$$

where $k_2 \geqslant 0$ is a constant, and we have used (2.7) and the fact of that

$$\int_{R^n\times R^{n\otimes d}} \left(|\overline{Y}|^2 + |\overline{Z}|^2\right) J(\overline{Y}, \overline{Z})\, d\overline{Y}\, d\overline{Z} < \infty.$$

Hence

$$E\left(\left|Y_{t\wedge\tau}^n - Y_{t\wedge\tau}^m\right|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau} \left|Z_s^n - Z_s^m\right|^2 ds + \frac{1}{2}\int_{t\wedge\tau}^{\tau} \left\|U_s^n - U_s^m\right\|^2 ds\right)$$
$$\leqslant k_2\int_t^{\infty}\int_{R^n} \left(u_1(s) + u_2(s)^2\right)\rho_1\left(E\left|Y_{s\wedge\tau}^n - Y_{s\wedge\tau}^m - \left(n^{-1} - m^{-1}\right)\overline{Y}\right|^2\right)$$
$$+ k_2\int_t^{\infty} \left(u_1(s) + u_2(s)^2\right) E\left|Y_{s\wedge\tau}^n - Y_{s\wedge\tau}^m\right|^2 ds + k_2\left(n^{-1} + m^{-1}\right).$$

It is easy to deduce by the Fatou lemma that

$$\limsup_{n,m\to\infty} E\left|Y_{t\wedge\tau}^n - Y_{t\wedge\tau}^m\right|^2 + \frac{1}{2}\limsup_{n,m\to\infty} E\int_{t\wedge\tau}^{\tau} \left(\left|Z_s^n - Z_s^m\right|^2 + \left\|U_s^n - U_s^m\right\|^2\right) ds$$
$$\leqslant k_2\int_t^{\infty} \left(u_1(s) + u_2(s)^2\right)\rho_2\left(\limsup_{n,m\to\infty} E\left|Y_{s\wedge\tau}^n - Y_{s\wedge\tau}^m\right|^2\right) ds,$$

where $\rho_2(u) = \rho_1(u) + u$. We then apply the Bahari's inequality to obtain that

$$\limsup_{n,m\to\infty} E\left|Y_{t\wedge\tau}^n - Y_{t\wedge\tau}^m\right|^2 = 0$$

and

$$\limsup_{n,m\to\infty} E\int_0^{\tau} \left(\left|Z_t^n - Z_t^m\right|^2 + \left\|U_t^n - U_t^m\right\|^2\right) dt = 0.$$

These, together with the BDG inequality, yield

$$\lim_{n,m\to\infty} E \sup_{0\leqslant t\leqslant\tau} \left|Y_t^n - Y_t^m\right|^2 = 0.$$

By the completeness of Banach space, we know that there exists a unique $(Y,Z,U) \in S^2_{\mathcal{F}}(0,\tau;R^n) \times L^2_{\mathcal{F}}(0,\tau;R^{n\otimes d}) \times H^2_{\mathcal{F}}(0,\tau;R^{n\otimes r})$ such that as $n\to\infty$,

$$E\sup_{0\leqslant t\leqslant\tau}\left|Y_t^n - Y_t\right|^2 \to 0,$$
$$E\int_0^\tau \left|Z_t^n - Z_t\right|^2 dt \to 0,$$
$$E\int_0^\tau \left\|U_t^n - U_t\right\|^2 dt \to 0.$$

Therefore we can take a subsequence $\{n_k\}$ of $\{n\}$, denote it by $\{n\}$ again such that almost surely for $(t,\omega)\in[0,\infty)\times\Omega$,

$$\left(Y^n_{t\wedge\tau}, Z^n_{t\wedge\tau}, U^n_{t\wedge\tau}\right) \to (Y_{t\wedge\tau}, Z_{t\wedge\tau}, U_{t\wedge\tau}) \quad \text{in } R^n\times R^{n\otimes d}\times L^2_\lambda\left(R^{n\otimes r}\right).$$

Hence by the continuity of f in (Y,Z,U), (2.5), (2.7), (H3) and the Lebesgue domination convergence theorem, we have that

$$E\int_0^\tau \left|f^n\left(t,Y_t^n,Z_t^n,U_t^n\right) - f(t,Y_t,Z_t,U_t)\right| dt \to 0, \quad n\to\infty.$$

It is easily seen that (Y,Z,U) is a solution of (1.1) by taking the limit on both sides of (2.6).

Step 2. Assume that (H1)–(H3) hold, and there exists a constant $k_0\geqslant 0$ such that $|\xi|\leqslant k_0$. In this case, similar to Corollary 1 in Situ [19], we define, for each natural number N,

$$f_1^N(t,Y,Z,U,\omega) = f_1(t,Y,Z,U,\omega)\cdot W^N(Y),$$
$$f_2^N(t,Y,Z,U,\omega) = f_2(t,Y,Z,U,\omega),$$

where $W^N(Y)\in C_0^\infty(R^n)$ is defined as

$$W^N(Y) = \begin{cases} 1 & \text{for } |Y|\leqslant N+2,\\ 0 & \text{for } |Y|\geqslant N+3,\end{cases}$$
$$\left|W^N(Y_1) - W^N(Y_2)\right| \leqslant \bar{k}_0|Y_1 - Y_2| \quad \text{for all } Y_1,Y_2\in R^n,$$

and satisfies $0\leqslant W^N(Y)\leqslant 1$, where $\bar{k}_0$ is a positive constant. It is not hard to check that $f^N(t,Y,Z,U)$ satisfies (H1)–(H3) and (2.5), but with the following

$$\bar{u}_1(t) = u_1(t)(1+N+3)(\bar{k}_0+1),$$
$$\bar{u}_2(t) = u_2(t),$$
$$\bar{\rho}(u) = \rho_1(u),$$

for (2.3) and (2.5). Then by step 1 and Lemma 2.2 there exists (Y^N,Z^N,U^N) to solve (1.1) with f^N as the coefficient. Moreover,

$$\left|Y^N_{t\wedge\tau}\right| \leqslant N_0+2 \quad \text{for all } N=1,2,\ldots.$$

Hence $(Y^{N_0},Z^{N_0},U^{N_0})$ is a solution of (1.1).

Step 3. Assume that (H1)–(H4) hold. Let $\xi^n = \xi\mathbf{1}_{|\xi|\leqslant n}$, then by step 2 there exists a triplet (Y^n,Z^n,U^n), which is a solution of following BSDE:

$$Y_t^n = \xi^n + \int_{t\wedge\tau}^\tau f\left(s,Y_s^n,Z_s^n,U_s^n\right) ds - \int_{t\wedge\tau}^\tau Z_s^n\, dW_s - \int_{t\wedge\tau}^\tau\int_E U_s^n(x)\widetilde{N}_K(ds,dx). \tag{2.8}$$

By Itô's formula and the BDG inequality, it is not hard to prove that (Y^n,Z^n,U^n) is a Cauchy sequence in $S^2_{\mathcal{F}}(0,\tau;R^n)\times L^2_{\mathcal{F}}(0,\tau;R^{n\otimes d})\times H^2_{\mathcal{F}}(0,\tau;R^{n\otimes r})$. Hence there exists a limit (Y,Z,U). Letting $n\to\infty$ on both sides of (2.8), we easily see that (Y,Z,U) is a solution of (1.1). □

Remark 2.1. In the case of $\tau \leqslant T$, $T > 0$, it is clear that Theorem 2.1 generalizes the corresponding result in Pardoux [12] or in Royer [17] by taking $u_1(t) = u_2(t) = K$, $K \geqslant 0$, for $t \leqslant T$ and $u_1(t) = u_2(t) = 0$ for $t > T$. When $\tau = \infty$ or $\tau < \infty$ a.s., it is known that Pardoux's existence and uniqueness result does not work, so Theorem 2.1 improves his result.

3. The comparison theorem of BSDEs with jumps

In this section we turn our attention to the comparison theorem of BSDEs with Poisson jumps. It should be noted that we need impose stronger assumptions on the coefficients than that of continuous BSDEs. Consider the following BSDEs ($n = 1$):

$$Y_t^1 = \xi^1 + \int_{t\wedge\tau}^{\tau} f^1\big(s, Y_s^1, Z_s^1, U_s^1\big)\,ds - \int_{t\wedge\tau}^{\tau} Z_s^1\,dW_s - \int_{t\wedge\tau}^{\tau}\int_E U_s^1(x)\widetilde{N}_K(ds,dx) \tag{3.1}$$

and

$$Y_t^2 = \xi^2 + \int_{t\wedge\tau}^{\tau} f^2\big(s, Y_s^2, Z_s^2, U_s^2\big)\,ds - \int_{t\wedge\tau}^{\tau} Z_s^2\,dW_s - \int_{t\wedge\tau}^{\tau}\int_E U_s^2(x)\widetilde{N}_K(ds,dx). \tag{3.2}$$

Assume that (H1)–(H4) hold for f^i and ξ^i, $i = 1,2$. Then from Theorem 2.1, (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) are the unique solutions of (3.1) and (3.2), respectively.

Theorem 3.1. *Suppose that f^i and ξ^i, $i = 1,2$, satisfy the following conditions with probability one*:

(1) $f^1(t,Y,Z,U) \leqslant f^2(t,Y,Z,U)$;
(2) $\xi^1 \leqslant \xi^2$;
(3) $f^1(t,Y,Z,U,\omega) = h(t,Y,Z,\omega) + \int_E c_t(x,\omega)U_t(x)^T\lambda(dx)$,

$$\big|h(t,Y_1,Z_1,\omega) - h(t,Y_2,Z_2,\omega)\big| \leqslant u_1(t)|Y_1 - Y_2| + u_2(t)|Z_1 - Z_2|,$$

where $\int_E |c_t(x,\omega)|^2\lambda(dx) \leqslant u_2(t)^2$ and $c_t^i(x,\omega)$, $i = 1,2,\dots,r$, is the ith component of $c_t(x,\omega)$ satisfying $c_t^i(x,\omega) \geqslant -1$ a.s.

Then, with probability one, we have

$$Y_t^1 \leqslant Y_t^2 \quad \textit{for all } t \in [0,\tau].$$

Before giving the proof of Theorem 3.1, we will provide an example similar to one chosen by Royer [17] to illustrate that condition (3) of Theorem 3.1 for f^1 with respect to U cannot be weakened as the usual Lipschitz condition.

Example 3.1. Consider two BSDEs driven by one-dimensional Poisson process with following coefficients:

$$f^1(t,Y,Z,U,\omega) = f^2(t,Y,Z,U) = u_2(t)\cdot\left[\int_{E_0}\big|U_t(x)\big|^2\lambda(dx)\right]^{\frac{1}{2}},$$

where $E_0 \subseteq E$, $0 < \lambda(E_0) < 1$; $u_2(t)$ is a function satisfying (2.1), but there exists $T > 0$ such that $u_2(t) = 1$ as $0 \leqslant t \leqslant T$.

It is easy to check by the Minkowski inequality that

$$\big|f^1(t,Y,Z,U_1) - f^1(t,Y,Z,U_2)\big| \leqslant u_2(t)\|U_1 - U_2\|.$$

Clearly, $(0,0,0)$ is the unique solution to BSDE (3.2) with $\xi^2 = 0$. If we set $\xi^1 = -\alpha N([0,T],E_0)$, $\alpha > 0$, then

$$(Y_t, Z_t, U_t) = \big(-\alpha N\big([0,t],E_0\big) + \alpha\big[\sqrt{\lambda(E_0)} - \lambda(E_0)\big](T-t), 0, -\alpha\mathbf{1}_{E_0}(x)\big)\cdot\mathbf{1}_{[t\leqslant T]}$$

is the unique solution to (3.1) with ξ^1 as the terminal value. However,

$$P(Y_t > 0) > 0 \quad \text{for any } t \in [0,T),$$

which contradicts the comparison theorem.

Proof of Theorem 3.1. Let $(\widehat{Y}_t, \widehat{Z}_t, \widehat{U}_t) := (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2, U_t^1 - U_t^2)$, then it satisfies the BSDE of following form:

$$\widehat{Y}_t = Y + \int_{t\wedge\tau}^{\tau} \left(a_s(\omega)\widehat{Y}_s + b_s(\omega)\widehat{Z}_s^T + \int_E c_s(x,\omega)\widehat{U}_s^T(x)\lambda(dx) + f_0(s,\omega) \right) ds - \int_{t\wedge\tau}^{\tau} \widehat{Z}_s\, dW_s - \int_{t\wedge\tau}^{\tau}\int_E \widehat{U}_s(x)\widetilde{N}_K(ds,dx), \tag{3.3}$$

where

$$\begin{aligned}
&Y = \xi^1 - \xi^2 \leqslant 0,\\
&a_t(\omega) = \mathbf{1}_{[\widehat{Y}_t \neq 0]}\big(h\big(t, Y_t^1, Z_t^1, \omega\big) - h\big(t, Y_t^2, Z_t^1, \omega\big)\big)(\widehat{Y}_t)^{-1},\\
&b_{1t}(\omega) = \mathbf{1}_{[\widehat{Z}_{1t} \neq 0]}\big(h\big(t, Y_t^2, Z_{1t}^1, \dots, Z_{dt}^1\big) - h\big(t, Y_t^2, Z_{1t}^2, \dots, Z_{dt}^1\big)\big)(\widehat{Z}_{1t})^{-1},\\
&b_{it}(\omega) = \mathbf{1}_{[\widehat{Z}_{it} \neq 0]}\big(h\big(t, Y_t^2, Z_{1t}^2, \dots, Z_{it}^1, \dots, Z_{dt}^1\big) - h\big(t, Y_t^2, Z_{1t}^2, \dots, Z_{it}^2, \dots, Z_{dt}^1\big)\big)\cdot(\widehat{Z}_{it})^{-1}, \quad i = 2, \dots, d,\\
&f_0(t,\omega) = f^1\big(t, Y_t^2, Z_t^2, U_t^2, \omega\big) - f^2\big(t, Y_t^2, Z_t^2, U_t^2, \omega\big) \leqslant 0.
\end{aligned}$$

We first assume that $c_t^i(x,\omega) > -1$ a.s., $i = 1, 2, \dots, r$. For any $0 \leqslant T < \infty$, set

$$\begin{aligned}
d\overline{P}_T &= \left[\exp\left(\int_0^T b_s dW_s - \frac{1}{2}\int_0^T |b_s|^2\, ds\right) \prod_{0<t\leqslant T}\prod_{i=1}^r \left(1 + \int_E c_t^i(x,\omega) N_{K_i}\big(\{t\}, dx\big)\right) \exp\left(-\sum_{i=1}^r \int_0^T \int_Z c_t^i(x,\omega)\lambda(dx)\, ds\right)\right] dP\\
&:= Z_T\, dP.
\end{aligned}$$

Then by Girsanov transformation theorem, there exists a probability measure $\overline{P}$ defined on the standard measurable space $(\Omega, \mathcal{F})$ such that $\overline{P}|_{\mathcal{F}_T} = \overline{P}_T$, and

$$\overline{W}_t = W_t - \int_0^t b_s(\omega)\, ds$$

is a Brownian motion under probability measure $\overline{P}$;

$$\overline{N}_K(dx,dt) = \widetilde{N}_K(dx,dt) - c_t(x,\omega)\lambda(dx)\, dt$$

is a $\overline{P}$-martingale measure. Hence (3.3) can be rewritten as:

$$\widehat{Y}_t = Y + \int_{t\wedge\tau}^{\tau} \big(a_s(\omega)\widehat{Y}_s + f_0(s,\omega)\big)\, ds - \int_{t\wedge\tau}^{\tau} \widehat{Z}_s\, d\overline{W}_s - \int_{t\wedge\tau}^{\tau}\int_E \widehat{U}_s(x)\overline{N}_K(ds,dx).$$

An application of Itô's formula yields that

$$\widehat{Y}_{t\wedge\tau} e^{\int_0^{t\wedge\tau} a_s(\omega)\, ds} = E^{\overline{P}}\big[\widehat{Y}_{t\wedge\tau} e^{\int_0^{t\wedge\tau} a_s(\omega)\, ds} \big| \mathcal{F}_{t\wedge\tau}\big] = E^{\overline{P}}\left[\left(e^{\int_0^\tau a_s(\omega)\, ds} Y + \int_{t\wedge\tau}^{\tau} e^{\int_0^s a_r(\omega)\, dr} f_0(s,\omega)\, ds\right)\middle| \mathcal{F}_{t\wedge\tau}\right] \leqslant 0,$$

This implies that $Y_t^1 \leqslant Y_t^2$, $\forall t \in [0,\tau]$, $\overline{P}$-a.s., and the required conclusion follows.

Assume that $c_t^i(x,\omega) \geqslant -1$, $i = 1, 2, \dots, r$. For this case, we can define

$$c_t^{in}(x,\omega) = c_t^i(x,\omega) + \frac{|c_t^i(x,\omega)|}{n}.$$

Then by Lemma 2.1, there exists a unique triplet $(\widehat{Y}_t^n, \widehat{Z}_t^n, \widehat{U}_t^n)$ solving the following BSDE:

$$\widehat{Y}_t^n = Y + \int_{t\wedge\tau}^{\tau} \left(a_s(\omega)\widehat{Y}_s^n + b_s(\omega)\big(\widehat{Z}_s^n\big)^T + \int_E c_s^n(x,\omega)\big(\widehat{U}_s^n\big)^T \lambda(dx) + f_0(s,\omega) \right) ds - \int_{t\wedge\tau}^{\tau} \widehat{Z}_s^n\, dW_s - \int_{t\wedge\tau}^{\tau}\int_E \widehat{U}_s^n(x)\widetilde{N}_K(ds,dx).$$

Noting that $c_t^{in}(x,\omega) > -1$, by above what have just proved, we immediately get

$$\widehat{Y}_t^n \leqslant 0, \quad \forall t \in [0,\tau].$$

Further, by the uniqueness of solution to (3.3), and letting $n \to \infty$, we have

$$Y_t^1 \leqslant Y_t^2, \quad \forall t \in [0,\tau]\ P\text{-a.s.}$$

The proof is complete. □

Remark 3.1. Checking the proof of Theorem 3.1, we easily find that if (1) and (2) hold, but $f^2(t,Y,Z,U,\omega)$ satisfies the condition (3), then Theorem 3.1 holds true.

Corollary 3.1. *Let* (1) *and* (2) *of Theorem* 3.1 *be satisfied. Assume further that* f^1 (*or* f^2) *satisfies the following conditions with probability one*:

$$\begin{aligned}&\left|f^1(t,Y,Z,U_1,\omega)-f^1(t,Y,Z,U_2,\omega)\right|\leqslant\left|\int_E c_t(x,\omega)\big(U_t^1(x)-U_t^2(x)\big)\lambda(dx)\right|,\\&\left|f^1(t,Y_1,Z_1,\omega)-f^1(t,Y_2,Z_2,\omega)\right|\leqslant u_1(t)|Y_1-Y_2|+u_2(t)|Z_1-Z_2|,\end{aligned}\tag{3.4}$$

where $c_t(x,\omega)$ *satisfies the same condition as that in Theorem* 3.1. *Then we have*

$$Y_t^1\leqslant Y_t^2\quad \text{for all } t\in[0,\tau] \text{ a.s.}$$

Proof. Similarly, $(\widehat{Y}_t,\widehat{Z}_t,\widehat{U}_t):=(Y_t^1-Y_t^2,Z_t^1-Z_t^2,U_t^1-U_t^2)$ satisfies the following BSDE:

$$\begin{aligned}\widehat{Y}_t=Y&+\int_{t\wedge\tau}^{\tau}\left(a_s(\omega)\widehat{Y}_s+b_s(\omega)\widehat{Z}_s^T+\int_E c_s(x,\omega)\widehat{U}_s^T(z)\lambda(dx)+f_0(s,\omega)\right)ds\\&-\int_{t\wedge\tau}^{\tau}\widehat{Z}_s\,dW_s-\int_{t\wedge\tau}^{\tau}\int_E\widehat{U}_s(x)\widetilde{N}_K(ds,dx),\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}&Y=\xi^1-\xi^2\leqslant 0,\\&a_t(\omega)=\mathbf{1}_{[\widehat{Y}_t\neq0]}\big(f^1(t,Y_t^1,Z_t^1,U_t^1)-f^1(t,Y_t^2,Z_t^1,U_t^1)\big)(\widehat{Y}_t)^{-1},\\&b_{1t}(\omega)=\mathbf{1}_{[\widehat{Z}_{1t}\neq0]}\big(f^1(t,Y_t^2,Z_{1t}^1,\dots,Z_{dt}^1,U_t^1)-f^1(t,Y_t^2,Z_{1t}^2,\dots,Z_{dt}^1,U_t^1)\big)(\widehat{Z}_{1t})^{-1},\\&b_{it}(\omega)=\mathbf{1}_{[\widehat{Z}_{it}\neq0]}\big(f^1(t,Y_t^2,Z_{1t}^2,\dots,Z_{it}^1,\dots,Z_{dt}^1,U_t^1)-f^1(t,Y_t^2,Z_{1t}^2,\dots,Z_{it}^2,\dots,Z_{dt}^1,U_t^1)\big)(\widehat{Z}_{it})^{-1},\quad i=2,3,\dots,d,\\&\hat{c}_t^1(z,\omega)=\mathbf{1}_{[\int_E c_t^1(x,\omega)\widehat{U}_{1t}(x)\lambda(dx)\neq0]}\big(f^1(t,Y_t^2,Z_t^2,U_{1t}^1,\dots,U_{rt}^1)-f^1(t,Y_t^2,Z_t^2,U_{1t}^2,\dots,U_{rt}^1)\big)\left(\int_E c_{1t}(x,\omega)\widehat{U}_{1t}(x)\lambda(dx)\right)^{-1},\\&\hat{c}_t^i(z,\omega)=\mathbf{1}_{[\int_E c_t^i(x,\omega)\widehat{U}_{it}(z)\lambda(dx)\neq0]}\big(f^1(t,Y_t^2,Z_t^2,U_{1t}^2,\dots,U_{it}^1,\dots,U_{rt}^1)\\&\qquad-f^1(t,Y_t^2,Z_t^2,U_{1t}^2,\dots,U_{it}^2,\dots,U_{rt}^1)\big)\left(\int_E c_t^i(x,\omega)\widehat{U}_{it}(x)\lambda(dx)\right)^{-1},\\&\tilde{c}_t^i(x,\omega)=\hat{c}_t^i(x,\omega)c_t^i(x,\omega),\qquad\left|\hat{c}_t^i(x,\omega)\right|\leqslant1,\quad i=1,2,\dots,r,\\&f_0(t,\omega)=f^1(t,Y_t^2,Z_t^2,U_t^2,\omega)-f^2(t,Y_t^2,Z_t^2,U_t^2,\omega)\leqslant0.\end{aligned}$$

Note that (3.5) has a unique zero solution from Lemma 2.1 when $Y=f_0=0$. Hence by Theorem 3.1, we have

$$Y_t^1\leqslant Y_t^2\quad\text{for all } t\in[0,\tau] \text{ a.s.}\qquad\square$$

Remark 3.2. It is easily seen from the proof of Corollary 3.1 that if (1) or (3) are replaced with $f^1(t,Y,Z,U)<f^2(t,Y,Z,U)$ or $\xi^1<\xi^2$, then $Y_t^1<Y_t^2$, $\forall t\in[0,\tau]$ a.s.

4. Some applications

4.1. The minimal solution of BSDE with Poisson jumps

This subsection is devoted to the applications of the comparison theorem for BSDEs with Poisson jumps. We will show an existence and uniqueness result for a BSDE with jumps and with continuous coefficient. Actually, the solution of such a BSDE is a minimal one.

Lemma 4.1. *Suppose that* $f:[0,\infty)\times R\times R^{1\otimes d}\times L_\lambda^2(R^{1\otimes r})\times\Omega\to R$ *is jointly continuous in* $(Y,Z,U)\in R\times R^{1\otimes d}\times L_\lambda^2(R^{1\otimes r})$ *with the properties of* (3.4) *and*

$$\big|f(t,Y,Z,U,\omega)\big| \leqslant u_1(t)\big(1+|Y|\big)+u_2(t)\big(|Z|+\|U\|\big). \tag{4.1}$$

Let

$$f^n(t,Y,Z,U,\omega)=\inf_{(X,\overline{Z})\in R\times R^d}\big\{f(t,X,\overline{Z},U,\omega)+nu_1(t)|X-Y|+nu_2(t)|\overline{Z}-Z|\big\},$$

then f^n is still $\mathcal{F}_t$-adapted and satisfies:

(1) $|f^n(t,Y,Z,U,\omega)| \leqslant u_1(t)(1+|Y|)+u_2(t)(|Z|+\|U\|)$, $f^n(t,Y,Z,U,\omega)\leqslant f^{n+1}(t,Y,Z,U,\omega)\leqslant f(t,Y,Z,U,\omega)$, $\forall n\in N$;
(2) $|f^n(t,Y_1,Z_1,U,\omega)-f^n(t,Y_2,Z_2,U,\omega)|\leqslant u_1(t)|Y_1-Y_2|+u_2(t)|Z_1-Z_2|$;
(3) (3.4) *still holds true*;
(4) *If* $(Y_n,Z_n,U_n)\to(Y,Z,U)$, *then* $f^n(t,Y_n,Z_n,U_n)\to f(t,Y,Z,U)$.

Proof. The adaptability of f^n is obvious. In analogy with Lemma 2 in [8], we can easily deduce that (1) and (2). (3) follows from (3.4) and the fact of

$$\big|f^n(t,Y,Z,U_1)-f^n(t,Y,Z,U_2)\big|\leqslant\sup_{X,\overline{Z}}\big|f(t,X,\overline{Z},U_1)-f(t,X,\overline{Z},U_2)\big|.$$

It suffices to show (4). Similar to Lemma 2 in [8], we can take a sequence $(X_n,\overline{Z}_n)$ such that

$$\begin{aligned}f^n(t,Y_n,Z_n,U_n)&\geqslant f(t,X_n,\overline{Z}_n,U_n)+nu_1(t)|X_n-Y_n|+nu_2(t)|\overline{Z}_n-Z_n|-\frac{1}{n}\\&\geqslant -u_1(t)\big(1+|Y_n|\big)-u_2(t)\big(|Z_n|+\|U_n\|\big)-\frac{1}{n}\\&\quad+(n-1)u_1(t)|X_n-Y_n|+(n-1)u_2(t)|\overline{Z}_n-Z_n|,\end{aligned} \tag{4.2}$$

which implies that

$$(n-1)u_1(t)|X_n-Y_n|+(n-1)u_2(t)|\overline{Z}_n-Z_n|\leqslant 2u_1(t)\big(1+|Y_n|\big)+2u_2(t)\big(|Z_n|+\|U_n\|\big)+\frac{1}{n}$$

and

$$\limsup_{n\to\infty} nu_1(t)|X_n-Y_n|+\limsup_{n\to\infty} nu_2(t)|\overline{Z}_n-Z_n|<\infty.$$

Therefore

$$\lim_{n\to\infty}X_n=Y,\qquad \lim_{n\to\infty}\overline{Z}_n=Z.$$

These together with (1) and (4.2) give (4), and the proof is complete. □

Theorem 4.1. *Under the assumptions of Lemma* 4.1 *for f, BSDE* (1.1) *has a minimal solution. Furthermore, if* (2.3) *is satisfied, then the solution is unique.*

Proof. We first consider the following BSDE:

$$\zeta_t=\xi+\int_{t\wedge\tau}^{\tau}h(s,\zeta_s,\eta_s,\theta_s,\omega)\,ds-\int_{t\wedge\tau}^{\tau}\eta_s\,dW_s-\int_{t\wedge\tau}^{\tau}\int_E\theta_s(x)\widetilde{N}_K(ds,dx), \tag{4.3}$$

where

$$h(t,\zeta,\eta,\theta,\omega)=u_1(t)\big[1+|\zeta|\big]+u_2(t)\big[|\eta|+\|\theta\|\big].$$

By Lemma 2.1, there exists a unique solution (ζ,η,θ) to (4.3). Next, we consider a sequence of BSDEs of following

$$Y_t^n=\xi+\int_{t\wedge\tau}^{\tau}f^n\big(s,Y_s^n,Z_s^n,U_s^n\big)\,ds-\int_{t\wedge\tau}^{\tau}Z_s^n\,dW_s-\int_{t\wedge\tau}^{\tau}\int_E U_s^n(x)\widetilde{N}_K(ds,dx). \tag{4.4}$$

Then by Lemmas 4.1 and 2.1, (4.4) has a unique solution. Furthermore, by Corollary 3.3, we have

$$Y_t^1\leqslant\cdots\leqslant Y_t^n\leqslant Y_t^{n+1}\leqslant\zeta_t\quad\text{a.s.}$$

It is not difficult to deduce by the BDG inequality that

$$E \sup_{0 \leqslant t \leqslant \tau} \left(\left| Y_t^1 \right|^2 + |\zeta_t|^2 \right) + E \int_0^\tau \left(\left| Q_t^1 \right|^2 + |\eta_t|^2 + \left\| P_t^1 \right\|^2 + \|\theta_t\|^2 \right) dt \leqslant C,$$

where C is a positive constant only depending on $\int_0^\infty (u_1(t) + u_2(t)^2)\, dt$ and $E|\xi|^2$.

Also

$$E \sup_n \sup_{0 \leqslant t \leqslant \tau} \left| Y_t^n \right|^2 \leqslant E \sup_{0 \leqslant t \leqslant \tau} \left(\left| Y_t^1 \right|^2 + |\zeta_t|^2 \right) \leqslant C. \tag{4.5}$$

Hence by monotone convergence theorem there exists a stochastic process $\{Y_t\}$ such that

$$\lim_{n \to \infty} Y_t^n = Y_t, \quad \forall t \in [0, \tau] \text{ a.s.}$$

Note that

$$E \int_0^\tau \left[u_1(t) + u_2(t)^2 \right] \left| Y_t^n - Y_t^m \right|^2 dt \to 0, \quad n, m \to \infty, \tag{4.6}$$

by the Fatou lemma. We further get by Itô's formula that

$$\begin{aligned} E \int_0^\tau \left| Z_t^n - Z_t^m \right|^2 dt + E \int_0^\tau \left\| U_t^n - U_t^m \right\|^2 dt &\leqslant 2E \int_0^\tau \left| Y_t^n - Y_t^m \right| \left| f^n\left(t, Y_t^n, Z_t^n, U_t^n\right) - f^m\left(t, Y_t^m, Z_t^m, U_t^m\right) \right| dt \\ &\leqslant \widehat{C} \left[\left(E \int_0^\tau u_1(t) \left| Y_t^n - Y_t^m \right|^2 dt \right)^{\frac{1}{2}} + \left(E \int_0^\tau u_2(t)^2 \left| Y_t^n - Y_t^m \right|^2 dt \right)^{\frac{1}{2}} \right], \end{aligned}$$

where $\widehat{C}$ is a positive constant, and we have used Holder's inequality and (4.5). Then there exists a pair $(Z, U) \in L_{\mathcal{F}}^2(0, \tau; R^{1 \otimes d}) \times H_{\mathcal{F}}^2(0, \tau; R^{1 \otimes r})$ such that

$$E \int_0^\tau \left| Z_t^n - Z_t \right|^2 dt + E \int_0^\tau \left\| U_t^n - U_t \right\|^2 dt \to 0, \quad n \to \infty. \tag{4.7}$$

Hence by the BDG inequality, we can easily obtain

$$E \sup_{0 \leqslant t \leqslant \tau} \left| Y_t^n - Y_t^m \right|^2 \to 0, \quad n, m \to \infty,$$

which implies that there exists a sequence of $\{n\}$ denoted by $\{n\}$ again such that

$$E \sup_{0 \leqslant t \leqslant \tau} \left| Y_t^n - Y_t \right|^2 \to 0, \quad n \to \infty. \tag{4.8}$$

Finally we need to prove that

$$E \int_t^\tau f^n\left(s, Y_s^n, Z_s^n, U_s^n\right) ds \to E \int_t^\tau f(s, Y_s, Z_s, U_s)\, ds, \quad n \to \infty. \tag{4.9}$$

By (4.7) we take a sequence $\{n_k\}$ of $\{n\}$ denoted by $\{n\}$ such that

$$E \int_0^\tau \left| Z_t^n - Z_t \right|^2 dt + E \int_0^\tau \left\| U_t^n - U_t \right\|^2 dt \leqslant \frac{1}{2^n},$$

then

$$E \int_0^\tau \left(\sup_n \left| Z_t^n - Z_t \right|^2 + \sup_n \left\| U_t^n - U_t \right\|^2 \right) dt \leqslant E \int_0^\tau \sum_n \left(\left| Z_t^n - Z_t \right|^2 + \left\| U_t^n - U_t \right\|^2 \right) dt < \infty. \tag{4.10}$$

Since

$$\left| f^n\left(t, Y_t^n, Z_t^n, U_t^n\right) \right| \leqslant u_1(t) \left[1 + \sup_n \left| Y_t^n \right| \right] + u_2(t) \left[\sup_n \left| Z_t^n \right| + \sup_n \left\| U_t^n \right\| \right]$$

and

$$\sup_n \big|Z_t^n\big|^2 + \sup_n \big\|U_t^n\big\|^2 \leqslant 2\Big(|Z_t|^2 + \sup_n \big|Z_t^n - Z_t\big| + \|U_t\|^2 + \sup_n \big\|U_t^n - U_t\big\|^2\Big),$$

then by (4.5), (4.10) and Lebesgue's dominated convergence theorem, (4.9) follows. Now taking limits in (4.4), we deduce that (Y, Z, U) is an adapted solution of (1.1).

Let $(\widehat{Y}, \widehat{Z}, \widehat{U})$ be an adapted solution of (1.1). By Corollary 3.1 we obtain that $Y^n \leqslant \widehat{Y}$, $\forall n$, and therefore $Y \leqslant \widehat{Y}$ proving that Y is the minimal solution of (1.1). If (2.3) is satisfied, then the uniqueness of solution comes from Theorem 2.1. □

4.2. Viscosity solutions of PDIEs

Feynman–Kac formula gives a probabilistic interpretation for linear second-order PDEs of elliptic or parabolic types, which has been generalized to the systems of semi-linear second-order PDEs by Peng [14], Pardoux and Tang [13], see also Darling and Pardoux [3] and references therein, with the help of BSDEs. This subsection can be seen as a continuation of such a theme, but here will give a probabilistic formula for a class of second-order partial differential and integral equations (PDIEs) of parabolic type by applying BSDEs with Poisson jumps.

For any $(t, x) \in [0, \infty) \times R^m$, we consider infinite horizon FBSDEs as follows ($t \leqslant s < \infty$):

$$X_s = x + \int_t^s b(r, X_r)\,dr + \int_t^s \sigma(r, X_r)\,dW_r + \int_t^s \int_E c(r, X_{r-}, e)\widetilde{N}_K(dr, de), \tag{4.11}$$

$$Y_s = h(X_\infty) + \int_s^\infty f(r, X_r, Y_r, Z_r, U_r)\,dr - \int_s^\infty Z_r\,dW_r - \int_s^\infty \int_E U_r(e)\widetilde{N}_K(dr, de). \tag{4.12}$$

Assume that all the coefficients in (4.11) and (4.12) are deterministic. In order to guarantee the existence and uniqueness of strong solution to (4.11), we assume that b, σ, c are all Lipschitz continuous with a Lipschitzian function satisfying (2.1). For (4.12) we will suppose that

(A1) $|h(X_1) - h(X_2)| \leqslant K|X_1 - X_2|$, where $K > 0$;
(A2) $|f(t, X, 0, 0, 0)| \leqslant u_1(t)[1 + |X|]$;
(A3) $|f(t, X, Y_1, Z_1, U) - f(t, X, Y_2, Z_2, U)| \leqslant u_1(t)|Y_1 - Y_2| + u_2(t)|Z_1 - Z_2|$;
(A4) Given an X, f satisfies (2.3) and (3.4).

Then by Theorem 4.1, (4.12) admits a unique solution. We denote the solution by $(Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})$, where $Y_r^{t,x}$ is $\mathcal{F}_r^t$-adapted, $(Z_r^{t,x}, U_r^{t,x})$ are $\mathcal{F}_r^t$-predictable, and $\mathcal{F}_r^t = \sigma(W_s - W_t, N_k((t,s], A), t \leqslant s \leqslant r, A \in \mathcal{B}(E)) \vee \mathcal{N}$. Obviously,

$$\big(u(t,x), \upsilon(t,x), \zeta(t,x)\big) := \big(Y_t^{t,x}, Z_t^{t,x}, U_t^{t,x}\big) \tag{4.13}$$

is deterministic. By the uniqueness of solution to (4.12), it is known that for any $t \leqslant s < \infty$,

$$Y_s^{t,x} = Y_s^{s, X_s^{t,x}} = u\big(s, X_s^{t,x}\big).$$

If there exists $u(t,x) \in C^{1,2}([0,\infty) \times R^m)$ solving the following parabolic type of partial differential and integral equation:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + (\mathcal{L}u)(t,x) = -f\big(t, x, u(t,x), \upsilon(t,x), \zeta(t,x)\big), \\ (\mathcal{L}u)(t,x) := \big\langle \nabla u(t,x), b(t,x)\big\rangle + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^m a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \int_E \big[u\big(t, x + c(t,x,e)\big) - u(t,x) - \big\langle \nabla u(t,x), c(t,x,e)\big\rangle\big]\lambda(de), \\ a_{ij}(t,x) := \big(\sigma\sigma^*(t,x)\big)_{ij}, \quad 1 \leqslant i, j \leqslant m, \\ \upsilon(t,x) := \big\langle \nabla u(t,x), \sigma(t,x)\big\rangle, \\ \zeta(t,x) := u\big(t, x + c(t,x,\cdot)\big) - u(t,x), \\ \lim\limits_{t\to\infty} u(t,x) := u(\infty, x) = h(x), \end{cases} \tag{4.14}$$

We have the following

Theorem 4.2. *Let assumptions* (A1)–(A4) *be satisfied. If* $u(t,x) \in C^{1,2}([0,\infty) \times R^m)$ *and* (u, υ, ζ) *solves PDIE* (4.14)*, then* (4.13) *holds, where* (X, Y, Z, U) *is the unique solution of* (4.11) *and* (4.12)*.*

Proof. Applying Itô's formula to $u(s, X_s^{t,x})$ on $[t, \infty)$, we have

$$
\begin{aligned}
u\big(\infty, X_\infty^{t,x}\big) = Y_\infty^{t,x} = h\big(X_\infty^{t,x}\big)
&= u(t,x) + \int_t^\infty \frac{\partial u(s, X_s^{t,x})}{\partial t}\, ds + \int_t^\infty \big\langle \nabla u\big(s, X_s^{t,x}\big), b\big(s, X_s^{t,x}\big)\big\rangle\, ds + \int_t^\infty \big\langle \nabla u\big(s, X_s^{t,x}\big), \sigma\big(s, X_s^{t,x}\big)\, dW_s\big\rangle \\
&\quad + \int_t^\infty \int_E \big[u\big(s, X_{s-}^{t,x} + c\big(s, X_{s-}^{t,x}, e\big)\big) - u\big(s, X_{s-}^{t,x}\big)\big]\widetilde{N}_K(ds, de) \\
&\quad + \int_t^T \int_E \big[u\big(s, X_{s-}^{t,x} + c\big(s, X_{s-}^{t,x}, e\big)\big) - u\big(s, X_{s-}^{t,x}\big) - \big\langle \nabla u\big(s, X_s^{t,x}\big), c\big(s, X_{s-}^{t,x}, e\big)\big\rangle\big]\lambda(de).
\end{aligned}
$$

On the other hand,

$$
Y_\infty^{t,x} = h\big(X_\infty^{t,x}\big) = u(t,x) - \int_t^\infty f\big(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}\big)\, ds + \int_t^\infty Z_s^{t,x}\, dW_s + \int_t^\infty \int_E U_s^{t,x}(e)\widetilde{N}_K(ds, de),
$$

which together with the uniqueness of decomposition for semi-martingale give the desired conclusion. □

Remark 4.1. (4.13) can be called a Feynman–Kac formula for PDIE (4.14). More generally, if (4.11) and (4.12) are coupled and have a unique solution, we can get the same Feynman–Kac formula as (4.13) for a PDIE, whose coefficients b, σ and c have the arguments $u(t,x)$, $\upsilon(t,x)$ and $\zeta(t,x)$.

We next exploit above Feynman–Kac formula and related comparison theorem for BSDEs with jumps to provide a probabilistic formula for the solution of quasi-linear parabolic PDIE. We shall prove in this subsection that the $u(t,x) := Y_t^{t,x}$ is a viscosity solution of the following backward quasi-linear second-order parabolic PDIE:

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} + (\mathcal{L}u)(t,x) + f\big(t, x, u(t,x), \upsilon(t,x), \zeta(t,x)\big) = 0, \\
&u(\infty, x) = h(x), \quad x \in R^m,\ 0 \leqslant t < \infty.
\end{aligned} \tag{4.15}
$$

Let us recall the definition of a viscosity solution for PDE (see Crandall, Ishii, and Lions [2]), we similarly give the definition of a viscosity solution for (4.15).

Definition 4.1. A continuous function $u(t,x)$ with $u(\infty, x) = h(x)$ is called a viscosity sub-solution (respectively sup-solution) of (4.15), if and only if $\forall \varphi(t,x) \in C^{1,2}([0,\infty) \times R^m)$ at the local minimum point (respectively maximum point) (t,x) of $\varphi - u$ has that

$$
\frac{\partial \varphi(t,x)}{\partial t} + (\mathcal{L}\varphi)(t,x) + f\big(t, x, u(t,x), \upsilon_\varphi(t,x), \zeta(t,x)\big) \geqslant 0 \quad (\text{respectively } \leqslant 0),
$$

where $\upsilon_\varphi(t,x) := \langle \nabla\varphi(t,x), \sigma(t, x, u(t,x))\rangle$. u is called a viscosity solutions of the PDIE (4.15) if it is both a viscosity sub-solution and sup-solution.

Theorem 4.3. *Assume that the coefficients in* (4.11) *are deterministic and satisfy Lipschitz condition. Under the assumptions of* (A1)–(A4)*, the function $u(t,x)$ defined by $Y_t^{t,x}$ is continuous, and is a viscosity solution of PDIE* (4.14)*.*

Proof. The continuity of $u(t,x)$ is similarly derived as Theorem 4.2 in [13]. We here only show that $u(t,x)$ is a viscosity sub-solution of the PIDE (4.14). The proof for viscosity sup-solution is similar. Let $\varphi \in C^{1,2}([0,\infty) \times R^m)$, and $(t,x) \in [0,\infty) \times R^m$ be the local minimum point of $\varphi - u$. Without loss of generality, we assume that $\varphi(t,x) = u(t,x)$. We now assume that

$$
\frac{\partial \varphi(t,x)}{\partial t} + (\mathcal{L}\varphi)(t,x) + f\big(t, x, u(t,x), \upsilon_\varphi(t,x), \zeta(t,x)\big) < 0,
$$

and we will find a contraction.

It follows from the above and the continuity of f, b, σ, c and φ that there exists $0 \leqslant \alpha < \infty$ such that for any $(s,y) \in [t,\infty) \times R^m$ satisfying $t \leqslant s \leqslant t + \alpha$, $|x - y| \leqslant \alpha$,

$$
\begin{aligned}
&u(s,y) \leqslant \varphi(s,y), \\
&\frac{\partial \varphi(s,x)}{\partial s} + (\mathcal{L}\varphi)(s,y) + f\big(s, y, u(s,y), \upsilon_\varphi(s,y), \zeta(s,y)\big) < 0.
\end{aligned}
$$

We now define

$$\tau := \inf\{s > t:\ |X_s^{t,x} - x| > \alpha\} \wedge (t + \alpha).$$

Let

$$(\overline{Y}_s, \overline{Z}_s, \overline{U}_s) := \left(Y_{s\wedge\tau}^{t,x}, \mathbf{1}_{[0,\tau]}(s) Z_s^{t,x}, \mathbf{1}_{[0,\tau]}(s) U_s^{t,x}\right), \quad t \leqslant s \leqslant t+\alpha,$$

then $(\overline{Y}_s, \overline{Z}_s, \overline{U}_s)$ is the solution of following BSDE:

$$\overline{Y}_s = u\left(\tau, X_\tau^{t,x}\right) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau)}(r) f\left(r, X_r^{t,x}, u\left(r, X_r^{t,x}\right), \overline{Z}_r, \overline{U}_r\right) dr - \int_s^{t+\alpha} \overline{Z}_r \, dW_r - \int_s^{t+\alpha}\int_E \overline{U}_r(e) \widetilde{N}_K(dr, de).$$

From Feynman–Kac formula (see Theorem 4.2, we still use the notion $\upsilon(t,x)$ and $\zeta(t,x)$ below), we have for $t \leqslant s \leqslant t+\alpha$,

$$Y_s = -\int_s^{t+\alpha} \mathbf{1}_{[0,\tau)}(r)\left[\frac{\partial\varphi(r, X_r^{t,x})}{\partial r} + (\mathcal{L}\varphi)\left(r, X_r^{t,x}\right)\right] dr + \varphi\left(\tau, X_\tau^{t,x}\right) - \int_s^{t+\alpha} \upsilon\left(r, X_r^{t,x}\right) dW_r - \int_s^{t+\alpha}\int_E \zeta\left(r, X_r^{t,x}, e\right) \widetilde{N}_K(dr, de).$$

From $u \leqslant \varphi$, and the choice of α and τ, we deduce that with the help of Corollary 3.1 that $\overline{Y}_t < Y_t$, namely, $u(t,x) < \varphi(t,x)$, which contradicts our assumption. □

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